

Anticommutator Norm Formula for Projection Operators

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ABSTRACT. We prove that for any two projection operators f, g on Hilbert space, their anticommutator norm is given by the formula

$$\|fg + gf\| = \|fg\| + \|fg\|^2.$$

The result demonstrates an interesting contrast between the commutator and anticommutator of two projection operators on Hilbert space. Specifically, the norm of the anticommutator $\|fg + gf\|$ is a simple quadratic function of the norm $\|fg\|$ while the commutator norm $\|fg - gf\|$ is not a function of $\|fg\|$. Nevertheless, the result gives the following bounds that are functions of $\|fg\|$ on the commutator norm: $\|fg\| - \|fg\|^2 \leq \|fg - gf\| \leq \|fg\|$.

1. The Main Result

The main result of this paper is proving the following norm formula.

Theorem 1.1. For any two projection operators f, g on Hilbert space,

$$\|fg + gf\| = \|fg\| + \|fg\|^2. \quad (1.1)$$

In particular, the anticommutator norm of projection operators is a simple quadratic function of the norm of their product. This is quite different from the commutator $fg - gf$ of projections since its norm is not a function of the norm of fg (see remark below), nor is $\|fg\|$ a function of $\|fg - gf\|$. In view of Theorem 1.1 we can nevertheless give bounds on the commutator norm that are functions of the norm $\|fg\|$. In this connection, the above theorem then has the following the consequence.

Corollary 1.2. For any two projection operators f, g on Hilbert space, one has

$$\|fg\| - \|fg\|^2 \leq \|fg - gf\| \leq \|fg\|.$$

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Proof. By Theorem 1.1 we have

$$\|fg - gf\| = \|2gf - (fg + gf)\| \geq 2\|gf\| - \|fg + gf\| \geq \|fg\| - \|fg\|^2.$$

Lemma 3.1 below gives the inequality $\|fg - gf\| \leq \|fg\|$. ■

(Lemma 3.1 and its proof are given at the end of the paper.)

We made an application of Theorem 1.1 in [2] in order to obtain sharp upper bound estimates for projection operator on Hilbert space that are nearly orthogonal to their (unitary) symmetries. Specifically, Theorem 1.1 has lead us to obtain the relatively larger bound of 0.455 that would guarantee that if a projection operator e and a Hermitian unitary operator w on Hilbert space satisfy $\|ewe\| < 0.455$, then a projection operator q exists such that¹ $qwq = 0$ and $\|e - q\| \leq \frac{1}{2}\|ewe\| + 4\|ewe\|^2$. (Further, q lies in the C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by e and wew^* .)

Remark 1.3. It is not hard to see that the commutator norm of projections $\|fg - gf\|$ is generally not a function of the norm $\|fg\|$. For instance, if $f = g \neq 0$, then $\|fg\| = 1$ and their commutator is zero, while if p, q are the two generating projections of the universal C*-algebra generated by two projections, then $\|pq\| = 1$ and $\|pq - qp\| = \frac{1}{2}$. Conversely, neither is the norm $\|fg\|$ a function of the norm $\|fg - gf\|$, since for the 2 by 2 matrix projections $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, one has $\|ab - ba\| = \frac{1}{2} = \|pq - qp\|$ while $\|ab\| = \frac{1}{\sqrt{2}} \neq 1 = \|pq\|$.

Remark 1.4. We caution that it is not enough to check equalities such as that in Theorem 1.1 for 2 by 2 matrices over the complex numbers and expect that they generally hold for all projections. For example, one can check that the equation

$$\|fg - gf\|^2 = \|fg\|^2(1 - \|fg\|^2) \tag{1.2}$$

holds for all projections in $M_2(\mathbb{C})$. One can, however, give simple examples of 4 by 4 projections for which equation (1.2) does not hold. Equation (1.2) also does not hold for the two projections p, q of the universal C*-algebra generated by two projections since they do not commute and $\|pq\| = 1$.

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¹The condition $qwq = 0$, of course, just means that q is orthogonal to its symmetric image wqw^* under w . And so the condition that $\|ewe\|$ is “small” means that e is nearly orthogonal to its symmetry.

2. Proof of $\|fg + gf\| \leq \|fg\| + \|fg\|^2$

We shall use the following lemma.

Lemma 2.1. For any two projections f, g and $m \geq 1$ we have $\|(fg)^m\| \leq \|fg\|^{2m-1}$.

Proof. By induction, one checks the equality $(fg)^m = (fgf)^{m-1}(fg)$ (for $m \geq 1$). Using $\|fgf\| = \|fg\|^2$, we have

$$\|(fg)^m\| \leq \|(fgf)^{m-1}\| \|fg\| \leq \|fg\|^{2m-2} \|fg\| = \|fg\|^{2m-1}$$

as required. ■

We begin by observing and establishing the following formula for the powers of the anticommutator in terms of polynomials in the operators fg, gf, fgf , and gfg :

$$(fg + gf)^n = P_n(fg) + P_n(gf) + Q_n(fgf) + Q_n(gfg) \quad (2.1)$$

where P_n, Q_n ($n = 1, 2, \dots$) are polynomials given recursively according to the dynamics

$$\begin{aligned} P_{n+1}(x) &= xP_n(x) + xQ_n(x), \\ Q_{n+1}(x) &= P_n(x) + xQ_n(x) \end{aligned} \quad (2.2)$$

with initial data $P_1(x) = x$, $Q_1(x) = 0$.² The equation (2.1) can be checked by induction by making strong use of the fact that f, g are projections. In order to find these polynomials in explicit form we express (2.2) in matrix form

$$\begin{bmatrix} P_{n+1} \\ Q_{n+1} \end{bmatrix} = \begin{bmatrix} x & x \\ 1 & x \end{bmatrix} \begin{bmatrix} P_n \\ Q_n \end{bmatrix}.$$

In order to telescope this expression we diagonalize the matrix here as follows:

$$\begin{bmatrix} x & x \\ 1 & x \end{bmatrix} = S \begin{bmatrix} x + \sqrt{x} & 0 \\ 0 & x - \sqrt{x} \end{bmatrix} S^{-1}, \quad S := \begin{bmatrix} \sqrt{x} & -\sqrt{x} \\ 1 & 1 \end{bmatrix}$$

(which is easily checked).

²Interestingly, these polynomials turn out to be similar to Fibonacci polynomials as they will be given in very similar closed forms in terms of \sqrt{x} .

Therefore, we calculate the polynomials as follows

$$\begin{aligned}
\begin{bmatrix} P_{n+1} \\ Q_{n+1} \end{bmatrix} &= \begin{bmatrix} x & x \\ 1 & x \end{bmatrix}^n \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = S \begin{bmatrix} (x+\sqrt{x})^n & 0 \\ 0 & (x-\sqrt{x})^n \end{bmatrix} S^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \\
&= \frac{1}{2\sqrt{x}} \begin{bmatrix} \sqrt{x} & -\sqrt{x} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (x+\sqrt{x})^n & 0 \\ 0 & (x-\sqrt{x})^n \end{bmatrix} \begin{bmatrix} 1 & \sqrt{x} \\ -1 & \sqrt{x} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \\
&= \frac{1}{2\sqrt{x}} \begin{bmatrix} \sqrt{x} & -\sqrt{x} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (x+\sqrt{x})^n & 0 \\ 0 & (x-\sqrt{x})^n \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} \\
&= \frac{1}{2\sqrt{x}} \begin{bmatrix} \sqrt{x} & -\sqrt{x} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(x+\sqrt{x})^n \\ -x(x-\sqrt{x})^n \end{bmatrix}
\end{aligned}$$

yielding the closed forms

$$\begin{aligned}
P_{n+1}(x) &= \frac{x}{2} \left[(x+\sqrt{x})^n + (x-\sqrt{x})^n \right], \\
Q_{n+1}(x) &= \frac{\sqrt{x}}{2} \left[(x+\sqrt{x})^n - (x-\sqrt{x})^n \right].
\end{aligned}$$

Next, we express these using their binomial expansions:

$$\begin{aligned}
(x+\sqrt{x})^n &= \sum_{j=0}^n \binom{n}{j} x^j x^{\frac{1}{2}(n-j)} \\
(x-\sqrt{x})^n &= \sum_{j=0}^n \binom{n}{j} x^j (-1)^{n-j} x^{\frac{1}{2}(n-j)}
\end{aligned}$$

Using the notation $\delta_2^k = \frac{1}{2}(1 + (-1)^k)$ which is 1 when k is even and 0 when k is odd, we can write

$$P_{n+1}(x) = \frac{1}{2}x \sum_{j=0}^n \binom{n}{j} x^j (1 + (-1)^{n-j}) x^{\frac{1}{2}(n-j)} = x \sum_{j=0}^n \binom{n}{j} x^j \delta_2^{n-j} x^{\frac{1}{2}(n-j)}.$$

Let us choose odd $n = 2N - 1$ so that

$$\begin{aligned}
P_{2N}(x) &= x \sum_{j=0}^{2N-1} \binom{2N-1}{j} x^j \delta_2^{2N-1-j} x^{\frac{1}{2}(2N-1-j)} \\
&= \sum_{j=0}^{2N-1} \binom{2N-1}{j} \delta_2^{j-1} x^{N+1+\frac{1}{2}(j-1)}
\end{aligned}$$

Now put $j = 2\ell - 1$ where $\ell = 1, 2, \dots, N$ to get

$$P_{2N}(x) = \sum_{\ell=1}^N \binom{2N-1}{2\ell-1} x^{N+\ell}.$$

Now we can compute the norm estimate at fg (or gf) as follows:

$$\|P_{2N}(fg)\| \leq \sum_{\ell=1}^N \binom{2N-1}{2\ell-1} \|(fg)^{N+\ell}\|.$$

Here we use the inequality $\|(fg)^m\| \leq \|fg\|^{2m-1}$ from Lemma 2.1 to get

$$\|P_{2N}(fg)\| \leq \sum_{\ell=1}^N \binom{2N-1}{2\ell-1} \|fg\|^{2N+2\ell-1} = \|fg\|^{2N-1} \sum_{\ell=1}^N \binom{2N-1}{2\ell-1} \|fg\|^{2\ell}$$

we note that the same bound is the same number for $\|P_{2N}(gf)\|$. At this juncture we make use of the identity

$$A_N(a) := \sum_{\ell=1}^N \binom{2N-1}{2\ell-1} a^{2\ell} = \frac{a}{2} [(1+a)^{2N-1} - (1-a)^{2N-1}]$$

which we shall call $A_N(a)$ for simplicity, where $a \geq 0$. We can then write

$$\|P_{2N}(fg)\| \leq \|fg\|^{2N-1} A_N(\|fg\|)$$

and we obtain

$$\|P_{2N}(fg)\| + \|P_{2N}(gf)\| \leq 2\|fg\|^{2N-1} A_N(\|fg\|).$$

Similarly we work out the norms $\|Q_{2N}(fgf)\|$ and $\|Q_{2N}(gfg)\|$.

$$Q_{n+1}(x) = \frac{\sqrt{x}}{2} \sum_{j=0}^n \binom{n}{j} x^j (1 - (-1)^{n-j}) x^{\frac{1}{2}(n-j)} = \sqrt{x} \sum_{j=0}^n \binom{n}{j} x^j \delta_2^{n-j-1} x^{\frac{1}{2}(n-j)}$$

and again inserting $n = 2N - 1$:

$$\begin{aligned} Q_{2N}(x) &= \sqrt{x} \sum_{j=0}^{2N-1} \binom{2N-1}{j} x^j \delta_2^{2N-1-j-1} x^{\frac{1}{2}(2N-1-j)} \\ &= \sum_{j=0}^{2N-1} \binom{2N-1}{j} \delta_2^j x^{N+\frac{j}{2}} \end{aligned}$$

put $j = 2\ell$ where $\ell = 0, 1, 2, \dots, N-1$:

$$Q_{2N}(x) = \sum_{\ell=0}^{N-1} \binom{2N-1}{2\ell} x^{N+\ell}.$$

The norm becomes (using $\|(fgf)^m\| \leq \|fg\|^{2m}$)

$$\|Q_{2N}(fgf)\| \leq \sum_{\ell=0}^{N-1} \binom{2N-1}{2\ell} \|(fgf)^{N+\ell}\| \leq \sum_{\ell=0}^{N-1} \binom{2N-1}{2\ell} \|fg\|^{2N+2\ell}$$

or

$$\|Q_{2N}(fgf)\| \leq \|fg\|^{2N} \sum_{\ell=0}^{N-1} \binom{2N-1}{2\ell} \|fg\|^{2\ell} = \|fg\|^{2N} B_N(a)$$

where we use the identity

$$B_N(a) := \sum_{\ell=0}^{N-1} \binom{2N-1}{2\ell} a^{2\ell} = \frac{1}{2} [(1+a)^{2N-1} + (1-a)^{2N-1}]$$

which we shall call $B_N(a)$ for convenience.

Let's write $a = \|fg\|$. Then we get

$$\begin{aligned} \|(fg + gf)^{2N}\| &= \|P_n(fg) + P_n(gf) + Q_n(fgf) + Q_n(gfg)\| \\ &\leq 2\|P_n(fg)\| + 2\|Q_n(fgf)\| \\ &\leq 2\|fg\|^{2N-1} A_N(\|fg\|) + 2\|fg\|^{2N} B_N(\|fg\|) \\ &= 2a^{2N-1} A_N(a) + 2a^{2N} B_N(a) \\ &= 2a^{2N-1} \cdot \frac{a}{2} [(1+a)^{2N-1} - (1-a)^{2N-1}] + 2a^{2N} \cdot \frac{1}{2} [(1+a)^{2N-1} + (1-a)^{2N-1}] \\ &= a^{2N} \cdot [(1+a)^{2N-1} - (1-a)^{2N-1}] + a^{2N} \cdot [(1+a)^{2N-1} + (1-a)^{2N-1}] \\ &= 2a^{2N} \cdot (1+a)^{2N-1}. \end{aligned}$$

Taking $2N$ -th roots,

$$\|fg + gf\| \leq 2^{1/2N} a (1+a)^{1-\frac{1}{2N}}$$

which in the limit as $N \rightarrow \infty$ gives

$$\|fg + gf\| \leq \|fg\| + \|fg\|^2.$$

3. Proof of $\|fg + gf\| \geq \|fg\| + \|fg\|^2$

Any two projection operators f, g on Hilbert space \mathcal{H} may be represented in matrix block forms as

$$f = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} D & V \\ V^* & D' \end{bmatrix}$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ where \mathcal{M} is the range of f (and \mathcal{M}^\perp that of $1-f$). Here, D, D' are positive operators on \mathcal{M} and \mathcal{M}^\perp , respectively, and $V : \mathcal{M}^\perp \rightarrow \mathcal{M}$, satisfying the relations

$$D - D^2 = VV^*, \quad DV + VD' = V, \quad D' - D'^2 = V^*V.$$

(in view of g being a projection). Since

$$fg(fg)^* = \begin{bmatrix} D & V \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ V^* & 0 \end{bmatrix} = \begin{bmatrix} D^2 + VV^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

we see that $\|fg\|^2 = \|D\|$. We now work out the powers of the anticommutator as follows. First, we have

$$fg + gf = \begin{bmatrix} D & V \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ V^* & 0 \end{bmatrix} = \begin{bmatrix} 2D & V \\ V^* & 0 \end{bmatrix}.$$

We observe that the powers of this anticommutator have the form

$$(fg + gf)^n = \begin{bmatrix} F_n(D) & F_{n-1}(D)V \\ \star & \star \end{bmatrix}$$

where $F_n(x)$ is a certain sequence of polynomials with integer coefficients – with initial data $F_1(x) = 2x, F_0(x) = 1$. We do not need to know the \star entries at the bottom of the matrix because we are interested in the northwestern³ block $F_n(D)$ of $(fg + gf)^n$. Multiplying

$$\begin{bmatrix} F_n(D) & F_{n-1}(D)V \\ \star & \star \end{bmatrix} \begin{bmatrix} 2D & V \\ V^* & 0 \end{bmatrix} = \begin{bmatrix} 2DF_n(D) + F_{n-1}(D)VV^* & F_n(D)V \\ \star & \star \end{bmatrix}$$

or

$$(fg + gf)^{n+1} = \begin{bmatrix} 2DF_n(D) + (D - D^2)F_{n-1}(D) & F_n(D)V \\ \star & \star \end{bmatrix}$$

from which we see that the polynomial sequence has the Fibonacci-type recursion relation

$$F_{n+1}(x) = 2xF_n(x) + (x - x^2)F_{n-1}(x).$$

Telescoping this as one would with ordinary Fibonacci numbers, one eventually gets

$$F_n(x) = \frac{1}{2}x^{n/2} [(\sqrt{x} + 1)^{n+1} - (\sqrt{x} - 1)^{n+1}].$$

(Indeed, one can check this by induction.) This polynomial function is (for each n) an increasing function over the interval $[0, \infty)$. Therefore, since D is a positive operator, we have

$$\|(fg + gf)^n\| \geq \|F_n(D)\| = F_n(\|D\|) = F_n(\|fg\|^2) = \frac{1}{2}\|fg\|^n [(\|fg\| + 1)^{n+1} - (\|fg\| - 1)^{n+1}]$$

³It is rather interesting in this case that the information regarding the anticommutator norm is contained in this block for large n .

$$= \frac{1}{2} \|fg\|^n (\|fg\| + 1)^{n+1} \left[1 - \left(\frac{\|fg\| - 1}{\|fg\| + 1} \right)^{n+1} \right].$$

Taking n -th roots (noting that the anticommutator $fg + gf$ is a Hermitian operator)

$$\|fg + gf\| \geq \frac{1}{2^{1/n}} \|fg\| (\|fg\| + 1)^{1+\frac{1}{n}} \left[1 - \left(\frac{\|fg\| - 1}{\|fg\| + 1} \right)^{n+1} \right]^{1/n}.$$

Letting $n \rightarrow \infty$ the right side converges to $\|fg\| + \|fg\|^2$ (since $(1 - c^n)^{1/n} \rightarrow 1$ for⁴ any $-1 < c < 1$).

This completes the proof of Theorem 1.1.

We end the paper with the proof of the lemma used by Corollary 1.2.

Lemma 3.1. For any two projections f, g on Hilbert space, $\|fg - gf\| \leq \|fg\|$. Further, $\|fg - gf\| = \|fg - fgf\|$.

Proof. Write

$$\|fg - gf\|^2 = \|(fg - gf)^*(fg - gf)\| = \|fgf + gfg - fgfg - gfgf\|$$

and note that the operator in the last norm can be written as the sum of two orthogonal positive operators:

$$fgf + gfg - fgfg - gfgf = fg(1 - f)gf + (1 - f)gfg(1 - f) = uu^* + u^*u$$

where $u = fg(1 - f)$. So its norm is the max of the norms of each term, both of which are equal to $\|u\|^2$. Thus,

$$\|fg - gf\| = \|u\| = \|fg - fgf\| = \|fg(1 - f)\| \leq \|fg\|$$

as needed. ■

Note added in proof. A generous colleague pointed out to the author that with a bit more work, one can deduce the anticommutator norm formula from a theorem of Halmos in [1]. Our proof, however, is self-contained and independent of this – and the formula (simple as it is) seems to be unknown.

⁴If $-1 < c < 1$ then $-1 < c^n \leq |c| < 1$ for each $n \geq 1$, which gives $0 < 1 - |c| \leq 1 - c^n < 2$ and the result follows by taking n -th roots.

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